Interaction of self-organized quasiparticles in a two-dimensional reaction-diffusion system: The formation of molecules

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In two-dimensional reaction-diffusion systems localized, solitary structures, that we call self-organized quasiparticles or spots, can be found as stable and stationary solutions. Combinations of two or more spots can lead to rather complex patterns, that can be understood by treating them as particles. These particles can interact with the boundaries of the system as well as with each other in different ways, that depend essentially on the parameters of the system. The interaction can be described by an approximation based on the exponential decay of the spots apart from their centers. The calculations reduce the dynamics of the system to some equations for the velocities of the spots. In particular, there is a parameter range where the interaction of two spots oscillates with their distance, which gives rise to infinitely many bounded states, resembling molecules. Investigating more than two spots molecules of numerous shapes have been obtained.
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I. INTRODUCTION

The understanding of complex dynamical systems is one of the most important subjects in modern physics. Due to their nonlinearity a great variety of biological (e.g., [1,2]), chemical (e.g., [3,4]), and physical (e.g., [5–7]) systems show very interesting phenomena of pattern formation. A typical example is the spontanous self-organization from a steady homogenous state to spatial or spatiotemporal patterns. Turing [8] proposed a two-component activator-inhibitor model that is able to describe the transition from a homogenous state to striped or hexagonal patterns on one- and two-dimensional domains. Mathematically these systems can be written as reaction-diffusion equations.

Small-amplitude structures occurring in these systems are well described by means of Ginzburg-Landau–type amplitude equations and its extensions (e.g., [9–12]). The description of large amplitude patterns can be performed by the analysis of solutions that are localized at least with respect to one spatial dimension such as stripes, spots [13], spheres, cylinders, etc. To obtain analytical results concerning existence and stability of these patterns the authors mostly consider certain asymptotical conditions of the parameters involved. Usually, characteristic time and/or length scales corresponding to the respective components have to be well separated to this end [14–18].

There are certain structures, though, that are systematically excluded by these approaches. One such class, which will be the topic of the present article, contains ensembles of spots, i.e., patterns localized in each direction that form bounded clusters resembling chemical molecules. Such objects have been observed experimentally (e.g., [19–22]) and numerically [20]. These patterns are ignored by the asymptotical approach referred to above as the second one. This is due to the interaction of neighboring spots, which, in the separated scales limit, is either repulsive, for large distances, or attractive if they are close to each other. Hence, stationary multispot patterns are unstable, and either merge or drift apart if they are not confined by the boundaries or by suitable inhomogeneities of the control parameters. There is an intermediate parameter region, though, which provides a different type of spot-spot interaction oscillating with their mutual distance that gives rise to infinitely many bounded states and, if more than two spots are involved, to many complex stable molecules. Some one-dimensional examples can be found in [13,23]. The notion of ''molecules'' reflects the analytical approach to investigate these structures.

In Sec. II the model equations are described and the existence and stability of localized stationary solutions is numerically proven. In Sec. III these are considered as fundamental particles similar to an atom in chemistry. Analytical investigations are restricted to the interaction of such particles. This leads to a dynamical system with one phase space dimension per particle and spatial dimension, i.e., a two-spot ensemble is described in a four-dimensional phase space in the framework of this article. The dynamical equations are based on certain coefficients that have to be determined numerically from the shape of the spots. This is sufficient to predict the shape and stability of many of the molecules observed in numerical simulations. Section IV summarizes some of these numerical results.

II. MODEL EQUATIONS AND BASIC SOLUTIONS

The investigated system is a two-component reaction-diffusion system on an infinite two-dimensional domain, which can be written as

$$\dot{v} = D_v \Delta v + f(v) - w + \kappa, \tag{1a}$$

$$\delta \dot{w} = D_w \Delta w + v - w. \tag{1b}$$

In these equations D_v and D_w are the diffusion coefficients of the activator $v = v(r) = v(x,y)$ and the inhibitor $w = w(r) = w(x,y)$ and $\Delta = \partial_{xx} + \partial_{yy}$ denotes the Laplacian in two dimensions. In the following the vector (v,w) will be abbreviated as u. The function $f(v)$ is a nonlinear, cubiclike

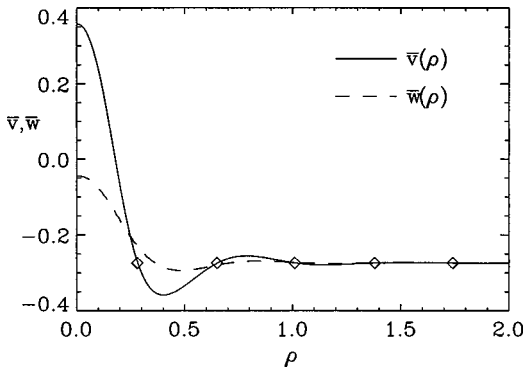

FIG. 1. Distribution of the components $\bar{v}(\rho)$ and $\bar{w}(\rho)$ of a localized, stable particlelike solution given the parameters $D_v=0.0028$, $D_w=0.025$, $\kappa=-0.1$, $\lambda=0.7$, and $\delta=1$. The intersection of $\bar{v}(\rho)$ with 0 is indicated by the markers.

function of v. The local dynamics is governed by the parameter κ and the function $f(v)$. For numerical simulations we choose $f(v)=\lambda v-v^3$ and Neumann boundary conditions. Parameters are always chosen such that the homogenous steady state $u_f=(v_f,w_f)$ is stable.

In this paper we want to investigate the behavior of stationary localized structures with rotational symmetry referred to as ''spots'' or ''quasiparticles.'' Using polar coordinates (ρ,φ) the solution describing a unique spot can be written as

$$\bar{u}(\rho)=(\bar{v}(\rho),\bar{w}(\rho)) \quad \text{with} \quad \lim_{\rho\to\infty}\bar{u}(\rho)=u_f. \tag{2}$$

An example for the components $\bar{v}(\rho)$ and $\bar{w}(\rho)$ of such an object is given in Fig. 1. Although it is not possible to predict the existence, shape and stability of these structures analytically, the numerical effort to obtain both the shape, and the stability information for a given set of parameters is rather low. Inserting (2) into Eqs. (1) yields the set of ordinary differential equations

$$0=D_v\left(\partial_{\rho\rho}\bar{v}+\frac{1}{\rho}\partial_\rho\bar{v}\right)+f(\bar{v})-\bar{w}+\kappa, \tag{3a}$$

$$0=D_w\left(\partial_{\rho\rho}\bar{w}+\frac{1}{\rho}\partial_\rho\bar{w}\right)+\bar{v}-\bar{w}. \tag{3b}$$

Since we demand a localized and smooth solution, the boundary conditions are $\partial_\rho\bar{u}(0)=0$ and $\bar{u}(\infty)=u_f$. Obviously $\bar{u}(\rho)=u_f$ is a trivial solution. For some parameters it is possible to obtain nontrivial solutions similar to those depicted in Fig. 1. Typically we get two solutions: In this case the one with lower amplitude is unstable and plays the role of a separatrix between the stationary state u_f and the second solution, which can be either stable or unstable. The stability of these structures can be investigated considering perturbations of the type

$$du_n(\rho,\varphi,t)=\sin(n\varphi+\phi_n)u_n(\rho,t), \quad n=0,1,2\ldots. \tag{4}$$

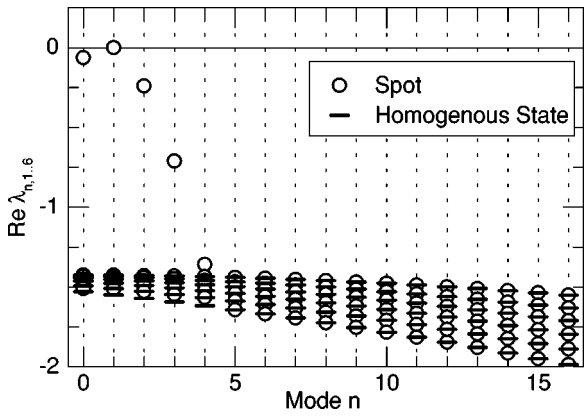

FIG. 2. Results of the numerical stability analysis for a stationary quasiparticle solution for the parameters $D_v=0.0025$, $D_w=0.025$, $\kappa=-0.32$, $\lambda=2$, and $\delta=1$. We show the six largest real parts of the eigenvalues λ_n, $n=1,\ldots,6$ of the linear stability analysis of the spot compared to those of the homogenous state. The zero eigenvalue for $n=1$ corresponds to the Goldstone mode of the quasiparticle.

This decomposition of possible perturbations reduces the two-dimensional stability problem to a number of one-dimensional problems of the form

$$\dot{v}_n=\left[D_v\left(\partial_{\rho\rho}+\frac{1}{\rho}\partial_\rho-\frac{n^2}{\rho^2}\right)+f'(\bar{v})\right]v_n-w_n, \tag{5a}$$

$$\dot{w}_n=\frac{1}{\delta}\left[D_w\left(\partial_{\rho\rho}+\frac{1}{\rho}\partial_\rho-\frac{n^2}{\rho^2}\right)-1\right]w_n+\frac{1}{\delta}v_n. \tag{5b}$$

Imposing continuity at $\rho=0$ leads to the boundary condition $u_n(0)=0$ for $n>0$. For $n=0$ we require smoothness and get $\partial_\rho u_n(0)=0$. To ensure the stability of the stationary solutions $\bar{u}(\rho)$ the real parts of the eigenvalues of the operator on the right-hand side have to be negative for all n. Typically solving the eigenvalue problem for $n=0,1,\ldots,10$ is sufficient to ensure the stability of the structure. In Fig. 2 we present the results of a stability analysis for the parameters

$$D_v=0.0025, \quad D_w=0.025, \quad \lambda=2, \quad \delta=1, \quad \kappa=-0.32. \tag{6}$$

Since we are able to prove the stability of the homogenous state analytically, it is useful to compare those results with a numerical stability analysis of the homogenous state. Figure 2 shows that only a few discrete eigenmodes are influenced by the shape of the spot and that for $n>5$ the results are nearly identical with the eigenvalues of the homogenous state. This suggests that modes with even higher n will not be unstable. For $n=1$ the stability analysis of the localized solution shows a zero eigenvalue that corresponds to the Goldstone modes of the spots $\partial_x\bar{u}$ and $\partial_y\bar{u}$.

Apart from their maximum the shape of the spots is governed by the linearization of Eq. (1). For increasing distance from the center it decays nearly exponentially towards the homogenous state. If the relation

$$-\frac{D_v}{D_w}-2\sqrt{\frac{D_v}{D_w}}<f'(v_f)<-\frac{D_v}{D_w}+2\sqrt{\frac{D_v}{D_w}} \tag{7}$$

is satisfied, the linearization yields solutions that decay exponentially, but oscillate around the stationary state. This type of decay can be seen in Fig. 1.

III. DERIVATION OF AN INTERACTION LAW

In the following we will derive an interaction law for the stable spots described above. The perturbation procedure applied is a two-dimensional extension of the technique utilized by Elphick, Meron, and Spiegel to treat the interaction of traveling pulses in one-dimensional systems [25]. We also found explicit representations of the relevant projectors in terms of the stationary solution such that we are able to derive the interaction in a quantitative manner. We start by shifting the offset of the components in order to map u_f to (0,0) by means of the transformation $u_{new}=u_{old}-u_f$. We will use u, v and w in this new meaning omitting the index, though. The system in Eq. (1) can be rewritten by splitting the right-hand side into linear and nonlinear terms. This yields the equation

$$\dot{u}=Lu+N(u), \tag{8}$$

where L is the linear operator

$$L=\begin{pmatrix} D_v\Delta+f'(v_f) & -1 \\ \frac{1}{\delta} & \frac{1}{\delta}(D_w\Delta-1) \end{pmatrix} \tag{9}$$

and N denotes the nonlinear operator

$$N(u)=\begin{pmatrix} f(v+v_f)-f'(v_f)v-f(v_f) \\ 0 \end{pmatrix}. \tag{10}$$

Now consider a superposition of an arbitrary number of spots $\overline{u_i}=\overline{u}(r-r_i)$ at different positions r_i. If all the distances between these quasiparticles are large enough, their superposition is a good approximation to the exact solution of Eq. (8) due to their localized shape. The error of this approximation scales with the distance between the spots. If all distances are increased simultaneously, the size of this error scales, according to the decay of the linearization of Eq. (1), like $\varepsilon\sim\exp(-\mu d_{min})/\sqrt{d_{min}}$, where d_{min} represents the smallest distance to be found between two members of a given ensemble. The coefficient μ corresponds to the absolute value of the real part of the eigenvalue λ, which can be obtained from the linearization of Eq. (1) as

$$\lambda^2=\frac{1}{2D_v}\left(\sigma\pm\sqrt{\sigma^2+4\frac{D_v}{D_w}[f'(v_f)-1]}\right), \tag{11}$$

with $\sigma=(D_v/D_w)-f'(v_f)$.

Since the system is homogenous there is no preferred position for a single stationary spot. Instead, there are two translational modes (Goldstone modes) and small perturbations can lead to a slow propagation. The overlap of two different spots can be interpreted as such a perturbation for each of them. Therefore, a reasonable approach for the description of the dynamics in this system is

$$u(r,\tau)=\sum_{i=1}^{N}\overline{u}(r-r_i(\tau))+\varepsilon s(r,\tau)=\sum_i\overline{u_i}+s. \tag{12}$$

In this equation the dynamics of the spots is assumed to appear on a slow time scale $\tau=\varepsilon t$, because the interaction of different quasiparticles is of $O(\varepsilon)$. $s(r,\tau)$ denotes the remaining error of this approach. Relaxation of the remaining degrees of freedom associated with the shape of the spots which happens on a faster time scale is considered to be finsished and is thus ignored. To get a unique decomposition (12) we have to add $2N$ further conditions. We choose

$$\langle\overline{u_{i,y}}|s\rangle=0 \quad \text{and} \quad \langle\overline{u_{i,x}}|s\rangle=0 \quad \text{for } i=1\ldots N, \tag{13}$$

where $\langle f\rangle$ denotes integration over the domain and an additional index, x represents the partial derivative $\partial/\partial x$. $\overline{u_{i,y}}$ and $\overline{u_{i,x}}$ are the Goldstone modes of a single spot and represent the two translational degrees of freedom: Adding the Goldstone mode to such a spot corresponds to a small spatial shift. Thus, we demand that $s(r,\tau)$ does not contribute to this shift.

The approach (12) satisfies Eq. (8) to order $O(1)$. The terms of $O(\varepsilon)$ yield the following equation:

$$\left[-L-\nabla N\left(\sum_i\overline{u_i}\right)\right]s=\frac{1}{\varepsilon}\left[N\left(\sum_i\overline{u_i}\right)-\sum_i N(\overline{u_i})\right]+\sum_i\nabla\overline{u_i}r_{i,\tau}. \tag{14}$$

The first term on the right-hand side is of $O(1)$, which can be proved by writing $N(v,w)$ as a power series.

To calculate the velocity $c_j=r_{j,\tau}$ of a given spot $\overline{u_j}$, it is necessary to eliminate the left-hand side of Eq. (14). For this purpose the equation is projected on the vectors

$$p_j=\left(-\frac{1}{\delta}\overline{v}_{j,x},\overline{w}_{j,x}\right) \quad \text{and} \quad q_j=\left(-\frac{1}{\delta}\overline{v}_{j,y},\overline{w}_{j,y}\right), \tag{15}$$

which are, to order ε, zero eigenvectors of the operator $[-L-\nabla N(\Sigma_i\overline{u_i})]^\dagger$, where $H^\dagger$ denotes the adjoint of a given operator H. Projection on p_j leads to an equation for the velocity $x_{j,t}$ of spot j in direction of the x axis. For the evaluation of the resulting equation it is useful to restrict the domain of integration to the disk $|r-r_j|<\frac{1}{2}\min_i(|r_i-r_j|)$, thus neglecting some terms of $O(\sqrt{\varepsilon})$. Returning to the original time scale t, this procedure leads to the result

$$\langle p_j|\overline{u}_{j,x}\rangle_\bigcirc x_{j,t}=-\sum_{i,i\neq j}\langle p_j|\nabla N(\overline{u_j})\overline{u_i}\rangle_\bigcirc+O(\varepsilon^{3/2}), \tag{16}$$

where $\langle\ldots\rangle_\bigcirc$ denotes the integration over the disk-shaped region. Projecting on q_j instead of p_j yields the velocity in direction of the y-axis.

Equation (16) provides a simple way to predict the movement of the spot j in the presence of the other quasiparticles. Thus, the reaction-diffusion system has been reduced to a

system of $2N$ ordinary differential equations, where N is the number of spots in the system. Since the velocity of a spot j depends only on its distance to the other spots in the system, the interaction of three or more of them can be constructed by a simple vector superposition, if their mutual interaction is known.

To obtain the interaction law for two spots, Eq. (16) can be written in simpler terms by means of the assumption, that both of them are located on the x axis at $(x_{(1,2)},y_{(1,2)})=(\pm d/2,0)$. In order to calculate the velocity of spot 1, the domain of integration can be extended to the right half-plane without losing accuracy. Afterwards it can be restricted to the y axis by means of Green's formula. Thus, the following equation for the speed $\dot{x}_1$ of the right spot as a function of the distance d to its neighbor is valid up to $O(\varepsilon^{3/2})$:

$$\dot{x}_1(d)=C(d)$$

$$=\frac{1}{\pi\int_0^\infty(\delta\overline{w}_{1,r}^2+\overline{v}_{1,r}^2)r\,dr}\int_{-\infty}^{\infty}[D_v(\overline{v}_{1,x}\overline{v}_{2,x}-\overline{v}_{1,xx}\overline{v}_2)$$

$$-D_w(\overline{w}_{1,x}\overline{w}_{2,x}-\overline{w}_{1,xx}\overline{w}_2)]dy|_{x=0}. \tag{17}$$

To evaluate Eq. (17) for arbitrary distances d, only the shape of a single spot $\overline{u}=(\overline{v},\overline{w})$ is needed. This shape can be computed with a very low numerical effort since we are studying objects with rotational symmetry and Eqs. (1) can be reduced to a spatially one-dimensional system.

After the function $\dot{x}_1(d)=C(d)$ has been calculated, the dynamics of N spots can be reduced to a system of equations of the form

$$\dot{r}_i=\sum_{\substack{j=1\\ j\neq i}}^{N}C(|r_i-r_j|)\frac{r_i-r_j}{|r_i-r_j|}+O(\varepsilon^{3/2}),\quad i=1,\ldots,N. \tag{18}$$

For all situations with more than two spots it has to be checked carefully, whether the absolute value of the sum exceeds the error estimate. If it does not, no prediction for the movement can be made. This problem will be addressed later, when we discuss some numerical results concerning the interaction of three and more spots.

IV. NUMERICAL RESULTS

To confirm Eqs. (17), (18), several numerical simulations have been carried out.

The numerical effort for calculating the interaction law from the approximation formula is negligible. On the other hand, the simulation of the full two-dimensional reaction-diffusion system needs considerable computational power. Since the velocity of the spots decreases almost exponentially with their distance dynamics almost vanishes as the distance grows. Furthermore the influence of the boundaries of the two-dimensional system on the spots must be taken into account. In these calculations Neumann boundary conditions of the form $n(x,y)\cdot\nabla u(x,y)=0$ with $(x,y)\in G$ have been used, where $n(x,y)$ denotes the normal vector of the boundary G of the system. For a rectangular domain, the influence of the boundaries can be interpreted in a simple way: They act as mirrors for the distribution $u=(v,w)$. To include this effect the interaction of the spots with their ''reflections'' at the boundaries of the system has to be taken care of. In order to calculate the interaction between two spots, one spot can be placed near a boundary. Thus the size of the rectangular domain can be reduced by a factor of 2. If no spot is closer than $\frac{3}{2}d_{max}$ to the domain boundary, assuming that d_{max} equals the largest distance between two interacting spots, the influences of the boundary are smaller than the error estimate and can be neglected.

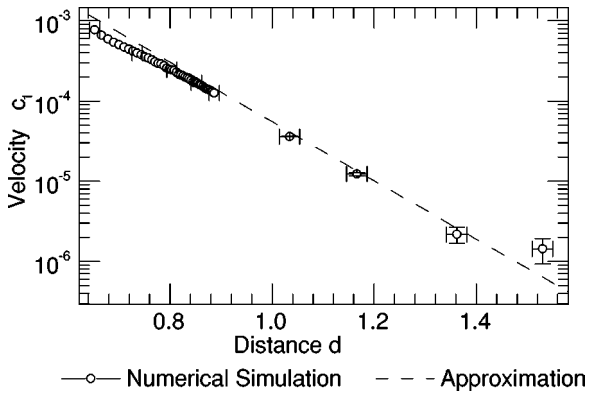

FIG. 3. Velocity c_1 of spot 1 as a function of the distance d to spot 2 for the parameters $D_v=0.0025$, $D_w=0.025$, $\kappa=-0.32$, $\lambda=2$, and $\delta=1$. The interaction is repulsive and decays exponentially with increasing distance d.

The exact form of the interaction depends directly on the shape of the stationary spots $\overline{u}(\rho)=(\overline{v}(\rho),\overline{w}(\rho))$. For large values of ρ these distributions only depend on the linearization of Eq. (8). This linear equation has two basic types of solutions: one that decays monotonously and another one that decays and oscillates in space with some wavelength k. The first type leads to repulsion in the relevant situations, while the second type exhibits domains of attraction and repulsion, depending on the distance between the spots.

A. Numerical results for two quasiparticles

The first results pertain to the interaction of two spots. For the numerical simulations the above mentioned set of parameters (6) has been used. In this case simple repulsion is expected. As an initial condition two spots were located at $y_{1,2}=0$ and $x_{1,2}=\pm d/2$, respectively.

In Fig. 3 the results of the numerical simulations are plotted together with the approximation derived from Eq. (17). For distances $d<0.9$ the velocities were obtained from a continous motion: The initial distance was $d=0.65$ and the spots travelled rather fast. For larger distances $d>0.9$, their movement becomes very slow, because the velocity depends exponentially on the distance. Thus it was necessary to determine the speed for some discrete spacings d. The comparison shows that the theoretical approximation is in rather good agreement with the numerical results. Of course, the approximation is only capable of capturing effects that lead to a movement of the spots. For geometries with $d<0.6$ the distinct spots lose their stability due to the strong overlap of the two distributions and a description in terms of a particle model is not possible. For this particular case the numerical

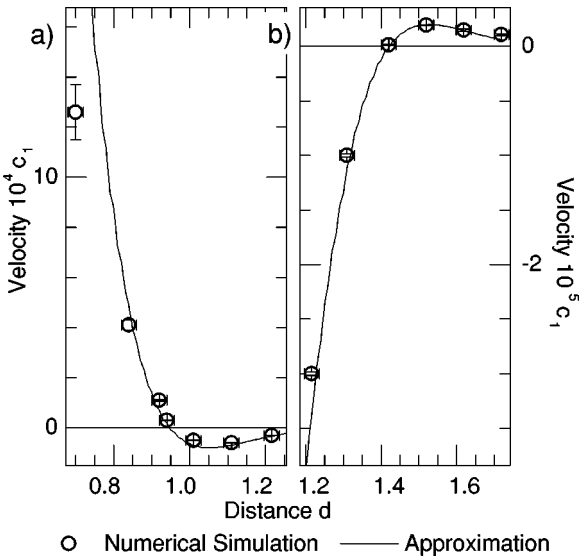

FIG. 4. Velocity of spot 1 as a function of the distance d to spot 2 for the parameters $D_v=0.0035$, $D_w=0.025$, $\kappa=-0.16$, $\lambda=1$, and $\delta=1$. (a) refers to $d<1.25$ and (b) to $d>1.25$. In this region of parameter space several discrete, stable distances $d_i=(0.95,1.86,2.77,\ldots)$ between two quasiparticles are possible.

results indicated that both spots disappear. If there are a large number of spots in a finite system, the mutual repulsion will lead to the formation of hexagonal patterns.

The case of oscillatory interaction was investigated for the parameters $D_v=0.0035$, $D_w=0.025$, $\kappa=-0.16$, $\lambda=1$, and $\delta=1$. Using the same geometry as in the previous case, we have obtained the results summarized in Fig. 4. The accordance between simulation and approximation is rather good. The most important feature of the system for this set of parameters is the existence of some discrete distances, for which two spots are bounded in a stationary, stable state. For the given parameters these distances are $d_i=(0.95,1.86,2.77,\ldots)$. This is due to the shape of the spots, which are surrounded by rings of alternating high and low activator concentrations. A detailed analysis of Eq. (17) reveals, that stationary, stable configurations correspond to the situation when the neighboring spots are located exactly on the rings with high activator concentration.

There is a difficulty that can arise in the case of oscillatory interaction. If the amplitude of the oscillatory rings is rather high, a superposition of two rings can lead to the ignition of new spots. This is due to the vicinity of the parameters to the Turing bifurcation. Of course, such an ignition of spots is not covered by Eq. (12).

B. Interaction of three or more quasiparticles

If there are more than two spots present in the system, it has to be checked, to what extent the two-particle approximation holds. The problem is that the size of ε is determined by the minimal distance between two spots. Since the unknown addend $\varepsilon s(r,\tau)$ in Eq. (12) scales with ε, its effect on more distant spots could become dominant.

In Fig. 5 some possible combinations of three or more spots for the case of oscillatory interaction are sketched. The parameters are the same as in Fig. 4 and the smallest distance

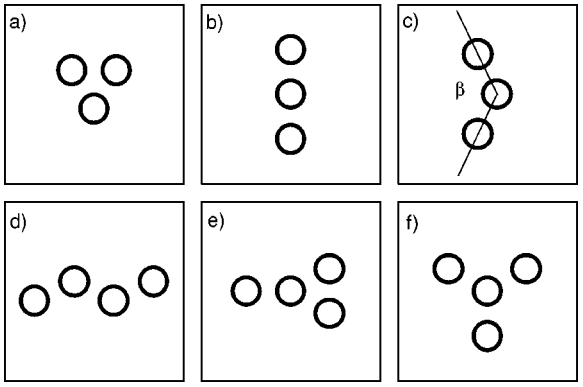

FIG. 5. Some stationary geometries for three or four spots for the parameters $D_v=0.0035$, $D_w=0.025$, $\kappa=-0.16$, $\lambda=1$, and $\delta=1$. The distance between two neighboring spots is $d_1=0.95$. Solutions (a), (c), (d), (e) and (f) are stable; only configuration (b) is unstable. The stability of the other solutions was proved by means of the analytical approximation or by numerical methods.

is equal to $d_1=0.95$ which is the first stationary, stable distance for two spots.

The configuration in Fig. 5(a) is stable, because the distances between all three particles are the same, and the size of ε is uniquely determined.

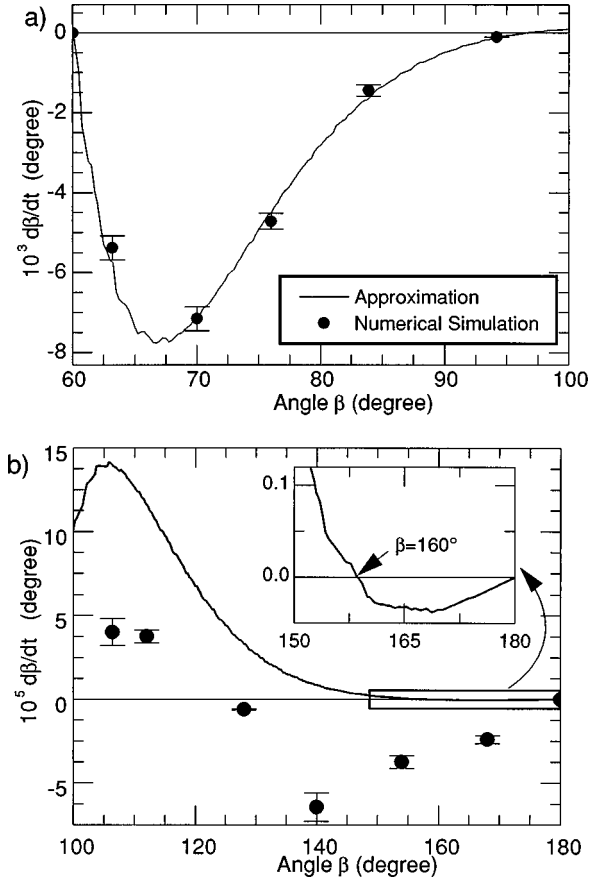

FIG. 6. Dynamics of the angular configuration of three spots from Fig. 4(c) as function of the angle β. The results as estimated by the theoretical approximation are compared with the numerical results for the parameters $D_v=0.0035$, $D_w=0.025$, $\kappa=-0.16$, $\lambda=1$, and $\delta=1$. (a) For small values of β the approximation is rather good. (b) For large β the results are qualitatively correct but suffer from the neglect of higher-order contributions.

In Fig. 5(b) a linear chain of three spots is sketched. The approximation fails in this case, because the effect of the outer spots onto each other is weaker than the effect of the unknown addend $s(r,\tau)$ from approach (12). The spot in the middle shields the mutual interaction of the outer particles. Due to the symmetry this configuration will be stationary but simulations showed that it is unstable.

Since the second stable distance $d_2=1.86$ is smaller than $2d_1=1.90$ it is likely that there exists an angular configuration corresponding to that in Fig. 5(c). There exists a strong and therefore fast interaction between the inner and outer spots. This leads to a fixed distance of $d_1=0.95$. The influence of the two outer spots onto each other should lead to a distance of about $d_2=1.86$, corresponding to an angle β of about 160°. The numerical proof of this assumption takes quiet a lot of computational time, since one has to make sure that the interaction between the inner and outer particles has completely vanished. After this process has ended the dynamics can be reduced to that of the angle β spanned by the three spots as depicted in Fig. 5(c).

The results of the numerics that comprise all three previous cases, Fig. 5(a)–5(c) are shown in Fig. 6. The approximation is valid only for small angles β, but does give a qualitatively correct picture of the dynamics: The linear chain of spots is dynamically unstable and there exists an angular configuration of three spots with an angle β of 125°. From the approximation an angle of $\beta=160°$ was estimated.

In Figs. 5(d)–5(f) there are some possible geometries with four quasiparticles: The stability for picture (d) can be concluded from the stability of the angular configuration (c). The stability of the two other configurations can only be guessed. However, numerical simulations show that both geometries are dynamically stable. Figure 7 shows the distribution of $v(x,y)$ resulting from the numerical simulation for the geometry of Fig. 5(f). The contour lines $v(x,y)=v_f$ emphasize the oscillatory behavior of the exponential decay.

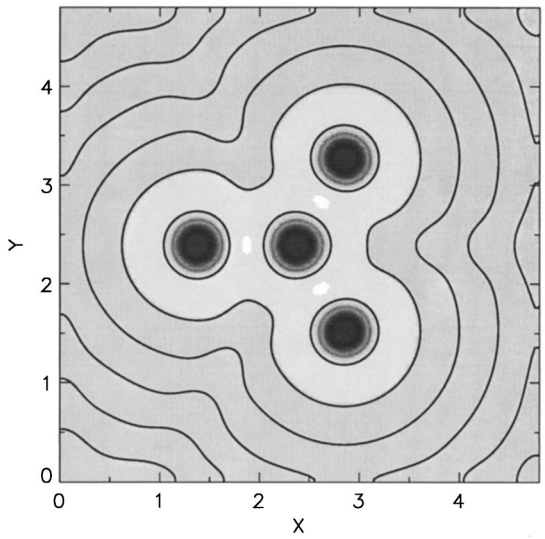

FIG. 7. Result of a numerical simulation for the geometry of Fig. 6(f). The gray-scale corresponds to values from -0.65 to 0.75 of $\overline{v}(x,y)$. The lines are given be $\overline{v}(x,y)=v_f$. They show that the solution decays in an oscillatory manner towards the homogenous state.

V. CONCLUSIONS

For parameter regions in which stationary spots are present, it was possible to determine their interaction and thus to calculate the resulting, stationary states.

A large variety of geometries can be constructed. Especially for parameters that allow oscillatory interaction in space, a lot of stationary, stable states can be found. These structures exhibit a strong resemblence to clusters of filaments in an dc gas-discharge system [24]. This system can be described by a set of two reaction-diffusion equations, which are similar to the model equations (1). Clusters of two bounded filaments and the configurations of Figs. 5(c) and 5(f) have been observed experimentally.

The same mathematical approach can be used for many types of localized patterns that can be constructed in reaction-diffusion systems. Elphick and co-workers [25,26] derived the asymptotics of an interaction law for propagating, localized pulses in a one-dimensional system. Wave trains of pulses and repulsion between these objects have been predicted theoretically. Experimentally these results have been verified using a chain of coupled nonlinear electric circuits [27]. Wave trains of up to four pulses have been observed. Other structures that could be used are spots in one-dimensional systems or stripes and moving stripes in two spatial dimensions.

Recent theoretical results [28] obtained for a set of three-component reaction-diffusion equations suggest that it is always possible to find parameters that lead to the propagation of any stationary structure that can be found in the corresponding two-component system. This leads to a variety of complex moving patterns, which is of special interest since it was possible to observe slowly moving clusters of filaments in an ac-driven gas-discharge experiment similar to that described in [19].

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